RESEARCH ARTICLE

Semiparametric Tests for Equality of Two Independent Variances

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ABSTRACT
In this paper, we propose new semiparametric and Bootstrap tests based on the ratio of two variances under a density ratio model and compare them with existing tests for testing the equality of two variances from two independent populations. We showed that the ratio of two independent variances follows an asymptotic log-normal distribution. Also presented in this paper are large sample asymptotic results of the semiparametric estimator of the ratio of the two sample variances. We also present a theoretical comparison of our semiparametric test on the ratio of two variances with the existing non-parametric test using Pitman's relative efficiency. Numerical comparisons of the performance of various tests are performed via simulation studies. We also use two numerical examples to illustrate the implementation of new tests.

KEYWORDS
Density ratio; empirical likelihood; variance ratio; Bootstrap test; goodness-of-fit; power analysis

1. Introduction

Testing homogeneity of populations has practical importance in real-world applications in that the variance measures the quality of products or performance of various processes in business, manufacturing industry, environmental science, healthcare services, etc. In statistics, some inferential procedures are heavily relying on the assumption of equal variances. If two independent populations are normally distributed with equal variances, the ratio of the two sample variances has an F distribution with certain degrees of freedom. Siegel and Tukey (1960) and Markowski and Markowski (1990) showed that the F test is extremely sensitive to the assumption of normal distribution.

For testing equality of variances from non-normal populations, there are also different tests available in the literature. The robust homogeneity tests of variances of multiple populations fall into two major categories: ANOVA-based F tests such as Levene (1960), Brown and Forsythe (1974), and many recent developments and modifications; and resampling based testing procedures such as those proposed by Boos and Brownie (1989), Lim and Loh (1996), Wludyka and Sa (2004), Charway and Bailer (2007), Parra-Frutos (2009) and Cahoy (2010), etc.

Due to its simplicity, robustness, and ease of implementation, Levene's test has been continuously cited, studied, and applied in a wide array of fields by researchers and practitioners since it was published. On the other hand, the ANOVA type F tests are approximately F distributed. Any serious violation of normality, homoscedasticity, and independence can significantly downgrade the performance of the test. Many researchers discussed the influence of various violations to the classical assumptions on tests for equal variances (See the recent work of Parra-Frutos, 2013, pages 1270-1271, for more information). Extensive applications of Levene's test and various modifications are summarized in Gastwirth et. al. (2009).

All ANOVA-based and Bootstrap homogeneous variance testing procedures can be naturally used for testing the equality of two independent population variances. Wan et al (2016) studied the difference of two population means and variances using a semiparametric density ratio model of Qin (1998) jointly with a semiparametric goodness-of-fit test (Qin and Zhang, 1997) and gave some asymptotic results on the testing procedures. In this paper, we use the same density ratio model as used in Wan et al (2016) to propose a homogeneous variance test based on the ratio of the two variances, an alternative to the well-known F ratio statistic. We showed that the variance ratio test statistic follows an asymptotic log-normal distribution under some conditions of the underlying density ratio model. We also propose a semiparametric

Bootstrap confidence interval-based test for the equality of two population variances that is suggested in Hall and Wilson (1991).

The rest of the paper is organized as follows. In Section 2, we outline the density ratio model and empirical likelihood-based semi-parametric estimation of the probability mass function. The sample variance will be defined as the weighted average of squared deviations from the semi-parametric mean. The asymptotic results of the proposed semi-parametric and non-parametric variance ratio test procedures will be presented in Sections 3 and 4 respectively. A theoretical comparison of the proposed semiparametric test and nonparametric variance ratio via Pitman's relative efficiency 5. The Bootstrap testing procedure will be given in Section 6. In Section 7, we present simulation studies on the power of the proposed tests and the existing tests including the one proposed by Wan et al (2016). Two numerical examples will be given in Section 8 and conclusions and recommendations will be given in Section 9.

2. Semiparametric Estimation of Variances

We consider the following two-sample density ratio model

$$\begin{aligned} &x_1, x_2, \cdots, x_{n_0} \sim g_0(x), \\ &y_1, y_2, \cdots, y_{n_1} \sim g_1(x) = \exp[\alpha + \beta^\tau r(x)] g_0(x) \end{aligned} \tag{1}$$

where (α, β^τ) is the vector of the regression coefficients, $\{x_i\}_{i=1}^{n_0}$ and $\{y_j\}_{j=1}^{n_1}$ are independent samples taken from $g_0(x)$ and $g_1(x)$, respectively. Assume $G_0(x)$ and $G_1(x)$ are the cumulative distribution of $g_0(x)$ and $g_1(x)$. The parameters in model (1) can be estimated from the two random samples.

There are different methods to estimate model parameters: probabilistic methods as discussed in Qin (1998), Bickel et al (2007), and Cheng and Chu (2004); and algorithmic methods as reviewed by Sugiyama et al (2010)). In this paper, we use the empirical likelihood-based probabilistic method proposed in Qin (1998) to estimate α, β^τ and probability distributions of $g_0(x)$ and $g_1(x)$.

To this end, we write $\{T_1, T_2, \cdots, T_n\} = \{x_1, x_2, \cdots, x_{n_0}, y_1, y_2, \cdots, y_{n_1}\}$. The random weight of observation T_i is defined to be $p_i = dG_0(T_i)$ for $i = 1, 2, \cdots, n$. These weights are considered as parameters of $g_0(x)$ and $g_1(x)$. The semiparametric likelihood function is given by

$$\begin{aligned} \mathcal{L}(\alpha, \beta, G_0) &= \prod_{i=1}^{n_0} dG_0(x_i) \prod_{j=1}^{n_1} \exp[\alpha + \beta^\tau r(y_j)] dG_0(y_j) \\ &= \prod_{i=1}^{n} p_i \prod_{j=1}^{n_1} \exp[\alpha + \beta^\tau r(y_j)] \end{aligned} \tag{2}$$

subject to constraints

$$\sum_{i=1}^{n} p_i = 1, \quad p_i \geq 0, \quad \sum_{i=1}^{n} p_i \{\exp[\alpha + \beta^\tau r(T_i)]\} = 1.$$

As shown in (Qin, 1998), for the fixed α and β^τ, the semiparametric estimator of

p_i is given by

$$\hat{p}_i = \frac{1}{n_0} \frac{1}{1 + \rho \exp[\alpha + \beta^\tau r(T_i)]}. \tag{3}$$

We substitute p_i with $\hat{p}_i$ in (2) and obtain the following profile empirical likelihood function of α and β^τ as follows

$$l(\alpha, \beta) = \sum_{j=1}^{n_1} [\alpha + \beta^\tau r(y_j)] - \sum_{i=1}^{n} \log\{1 + \rho \exp[\alpha + \beta^\tau r(T_i)]\} - n \log n_0. \tag{4}$$

where $\rho = n_1/n_0$. The maximum empirical likelihood estimate of α and β^τ, denoted by $\tilde{\alpha}$ and $\tilde{\beta}^\tau$ is the solution to the following system of score equations

$$\begin{aligned} \frac{\partial l(\alpha, \beta)}{\partial \alpha} &= n_1 - \sum_{i=1}^{n} \frac{\rho \exp[\alpha + \beta^\tau r(T_i)]}{1 + \rho \exp[\alpha + \beta^\tau r(T_i)]} = 0, \\ \frac{\partial l(\alpha, \beta)}{\partial \beta} &= \sum_{j=1}^{n_1} r(y_j) - \sum_{i=1}^{n} \frac{\rho \exp[\alpha + \beta^\tau r(T_i)]}{1 + \rho \exp[\alpha + \beta^\tau r(T_i)]} r(T_i) = 0. \end{aligned} \tag{5}$$

We substitute the semiparametric regression coefficients with $\tilde{\alpha}$ and $\tilde{\beta}^\tau$ and obtain the semiparametric estimator of the cell probability under $G_0(t)$ as follows

$$\tilde{p}_i = \frac{1}{n_0} \frac{1}{1 + \rho \exp[\tilde{\alpha} + \tilde{\beta}^\tau r(T_i)]}. \tag{6}$$

The cell weights under $G_1(t)$ is given by

$$\tilde{q}_i = \frac{1}{n_0} \frac{\exp[\tilde{\alpha} + \tilde{\beta}^\tau r(T_i)]}{1 + \rho \exp[\tilde{\alpha} + \tilde{\beta}^\tau r(T_i)]}. \tag{7}$$

2.1. Semiparametric Empirical Likelihood Estimate of Variances of $G_0(t)$ and $G_1(t)$

Using $\tilde{\alpha}$ and $\tilde{\beta}$ obtained from (5), the semiparametric empirical likelihood estimators of $G_0(t)$ and $G_1(t)$, denoted by $\tilde{G}_0(t)$ and $\tilde{G}_1(t)$, are given respectively by

$$\begin{aligned} \tilde{G}_0(t) &= \sum_{i=1}^{n} \tilde{p}_i I[T_i \le t] = \frac{1}{n_0} \sum_{i=1}^{n} \frac{I[T_i \le t]}{1 + \rho \exp[\tilde{\alpha} + \tilde{\beta}^\tau r(T_i)]}, \\ \tilde{G}_1(t) &= \sum_{i=1}^{n} \tilde{q}_i I[T_i \le t] = \frac{1}{n_0} \sum_{i=1}^{n} \frac{\exp[\tilde{\alpha} + \tilde{\beta}^\tau r(T_i)] I[T_i \le t]}{1 + \rho \exp[\tilde{\alpha} + \tilde{\beta}^\tau r(T_i)]}. \end{aligned} \tag{8}$$

where $I[\cdot]$ is the indicator function. Denote $\tilde{w}(x) = \exp[\tilde{\alpha} + \tilde{\beta}^\tau r(x)]$, that is, $\tilde{w}_i =$

$\exp[\tilde{\alpha}+\tilde{\beta}^\tau r(T_i)]$. with these notations, we write $\tilde{p}_i = d\tilde{G}_0(T_i)$ and $\tilde{q}_i = \tilde{p}_i\tilde{w}_i = d\tilde{G}_1(T_i)$. We then write the semiparametric estimator of the two population variances as follows

$$\begin{aligned}
\tilde{\mu}_0 &= \sum_{i=1}^{n} \tilde{p}_i T_i, \quad \tilde{\mu}_1 = \sum_{i=1}^{n} \tilde{q}_i T_i, \\
\tilde{\sigma}_0^2 &= \sum_{i=1}^{n} \tilde{p}_i T_i^2 - \left(\sum_{i=1}^{n} \tilde{p}_i T_i\right)^2, \\
\tilde{\sigma}_1^2 &= \sum_{i=1}^{n} \tilde{q}_i T_i^2 - \left(\sum_{i=1}^{n} \tilde{q}_i T_i\right)^2.
\end{aligned} \tag{9}$$

Remark: We can see from (8) that the ratio of sample sizes $\rho = n_1/n_0$ impacts the estimate of $G_0(t)$ and $G_1(t)$, hence, $\tilde{\sigma}_0^2$ and $\tilde{\sigma}_1^2$. We will make some suggestions on the sampling plans for obtaining better estimates of the variance in later sections.

2.2. Nonparametric Estimate of the Variance of $G_0(t)$ and $G_1(t)$

In contrast to the semiparametric empirical likelihood estimator of $G_0(t)$ and $G_1(t)$, the standard nonparametric empirical distributions are given by

$$\hat{G}_0 = \frac{1}{n_0}\sum_{i=1}^{n_0} I[x_i \le t] \quad \text{and} \quad \hat{G}_1 = \frac{1}{n_1}\sum_{j=1}^{n_1} I[y_j \le t] \tag{10}$$

respectively. The corresponding estimator of the population variances of the two independent populations are given by

$$\hat{\sigma}_0^2 = \frac{1}{n_0}\sum_{i=1}^{n_0}(x_i - \bar{x})^2 \quad \text{and} \quad \hat{\sigma}_1^2 = \frac{1}{n_1}\sum_{j=1}^{n_1}(y_j - \bar{y})^2. \tag{11}$$

The test statistic for equality of two variances and the related asymptotic results will be discussed in the next section.

3. Semiparametric Variance Ratio and Asymptotic Results

Wan et al (2016) use the difference between the two estimators of variances to test the equality of the two population variances. In this paper, we test the equality of the two population variances using the ratio of the two variances. This method is considered as a semiparametric extension of the well-known F test on the ratio of two variances from normal populations.

3.1. Notations

We will use the same notations that are used in Wan et al (2016) to present some asymptotic results in the following. Let (α_0, β_0) be the true value of (α, β) in the density ratio model (1) and denote $w(x) = \exp[\alpha_0 + \beta_0^\tau r(x)]$. We also assume that $\lim_{n\to\infty} n_1/n_0 = \lim_{n\to\infty} \rho(n)$ exists. For $k = 1, 2, 3$. Using indicator function $I[\cdot]$, we denote $I_0(k) = I[k > 0]$, $I_1(k) = I[k = 1]$, and $I_2(k) = I[k = 2]$. The following notations will be used to discuss the asymptotic properties of both non-parametric and semiparametric tests for equal variances.

$$\begin{aligned} A_k &= \int_{-\infty}^{\infty} \frac{w(y)}{1+\rho w(y)} [r(y)]^{I_0(k)} [r^\tau(y)]^{I_2(k)} dG_0(y), \\ C_k &= \int_{-\infty}^{\infty} \frac{w(y)}{1+\rho w(y)} (y-\mu_0)^{2[1+I_2(k)]} [r(y)]^{I_1(k)} dG_0(y), \\ C'_k &= \int_{-\infty}^{\infty} \frac{w(y)}{1+\rho w(y)} (y-\mu_1)^{2[1+I_2(k)]} [r(y)]^{I_1(k)} dG_0(y), \\ C''_2 &= \int_{-\infty}^{\infty} \frac{w(y)}{1+\rho w(y)} (y-\mu_0)^2 (y-\mu_1)^2 dG_0(y). \end{aligned} \tag{12}$$

The corresponding point estimators are given in the following.

$$\begin{aligned} \tilde{A}_k &= \sum_{i=1}^{n} \frac{\tilde{w}(T_i)}{1+\rho\tilde{w}(T_i)} [r(T_i)]^{I_0(k)} [r^\tau(T_i)]^{I_2(k)} \tilde{p}_i, \\ \tilde{C}_k &= \sum_{i=1}^{n} \frac{\tilde{w}(T_i)}{1+\rho\tilde{w}(T_i)} (T_i-\tilde{\mu}_0)^{2[1+I_2(k)]} [r(T_i)]^{I_1(k)} \tilde{p}_i, \\ \tilde{C}'_k &= \sum_{i=1}^{n} \frac{\tilde{w}(T_i)}{1+\rho\tilde{w}(T_i)} (T_i-\tilde{\mu}_1)^{2[1+I_2(k)]} [r(T_i)]^{I_1(k)} \tilde{p}_i, \\ \tilde{C}''_2 &= \sum_{i=1}^{n} \frac{\tilde{w}(T_i)}{1+\rho\tilde{w}(T_i)} (T_i-\tilde{\mu}_0)^2 (T_i-\tilde{\mu}_1)^2 \tilde{p}_i. \end{aligned} \tag{13}$$

The semiparametric estimators $\tilde{\mu}_0$ and $\tilde{\mu}_1$ are given in (9). Furthermore, we denote

$$A = \begin{pmatrix} A_0 & A_1^\tau \\ A_1 & A_2 \end{pmatrix}, \quad S = \frac{\rho}{1+\rho} A \quad \text{and} \quad \tilde{A} = \begin{pmatrix} \tilde{A}_0 & \tilde{A}_1^\tau \\ \tilde{A}_1 & \tilde{A}_2 \end{pmatrix}. \tag{14}$$

3.2. Some Asymptotic Results

We now present several theorems to establish the asymptotic results of the semiparametric estimator of the variance ratio. The first theorem concerns the asymptotic distribution of the semiparametric estimators of the two population variances under model (1).

Theorem 3.1. *If A^{-1} exists, under model (1), we have the following asymptotic results*

$$\sqrt{n} \begin{pmatrix} \tilde{\sigma}_0^2 - \sigma_0^2 \\ \tilde{\sigma}_1^2 - \sigma_1^2 \end{pmatrix} \xrightarrow{d} N\left(0, \Sigma_{semi}\right) \tag{15}$$

where

$$\Sigma_{semi} = \begin{pmatrix} \Sigma_{00}^s & \Sigma_{01}^s \\ \Sigma_{10}^s & \Sigma_{11}^s \end{pmatrix}$$

and

$$\begin{aligned}
\Sigma_{00}^s =& (1+\rho)\left\{\int_{-\infty}^{\infty}(u-\mu_0)^4 dG_0(u) - \left[\int_{-\infty}^{\infty}(u-\mu_0)^2 dG_0(u)\right]^2\right\} \\
&- \rho(1+\rho)\left\{C_2 - (C_0, C_1^\tau)A^{-1}\begin{pmatrix} C_0 \\ C_1 \end{pmatrix}\right\}. \\
\Sigma_{11}^s =& \frac{1+\rho}{\rho}\left\{\int_{-\infty}^{\infty}(u-\mu_1)^4 dG_1(u) - \left[\int_{-\infty}^{\infty}(u-\mu_1)^2 dG_1(u)\right]^2\right\} \\
&- \frac{1+\rho}{\rho}\left\{C_2' - (C_0', C_1'^\tau)A^{-1}\begin{pmatrix} C_0' \\ C_1' \end{pmatrix}\right\}. \\
\Sigma_{01}^s =& [\Sigma_{10}^s]^\tau = (1+\rho)\left\{C_2'' - (C_0, C_1^\tau)A^{-1}\begin{pmatrix} C_0' \\ C_1' \end{pmatrix}\right\}.
\end{aligned} \tag{16}$$

Proof. The proof of the above theorem is straightforward. See the proof of Theorem 3.1 in Wan et al (2016) for derivations of the major components in the proof. □

Next, we present the asymptotic result of the semiparametric estimator of the ratio of the two sample variances.

Theorem 3.2. *Let $U = (1/\sigma_0^2, -1/\sigma_1^2)^\tau$ and σ_0^2 and σ_1^2 be the true values of the two population variances. Under the same condition of Theorem 3.1, we have*

$$\sqrt{n}\left[\log\left(\tilde{\sigma}_0^2/\tilde{\sigma}_1^2\right) - \log\left(\sigma_0^2/\sigma_1^2\right)\right] \xrightarrow{d} \mathcal{N}\left(0, \Omega_{semi}\right) \tag{17}$$

where $\Omega_{semi} = U^\tau \Sigma_{semi} U$. Σ_{semi} is explicitly specified in Theorem 3.1.

Proof. We perform the first-order Taylor expansion on $\log(\tilde{\sigma}_0^2/\tilde{\sigma}_1^2)$ at (σ_0^2, σ_1^2) and obtain

$$\begin{aligned}
\log(\tilde{\sigma}_0^2/\tilde{\sigma}_1^2) =& \log(\sigma_0^2/\sigma_1^2) + \frac{1}{\sigma_0^2}(\tilde{\sigma}_0^2 - \sigma_0^2) - \frac{1}{\sigma_1^2}(\tilde{\sigma}_1^2 - \sigma_1^2) + o_p(\Delta) \\
=& \log(\sigma_0^2/\sigma_1^2) + \left(\frac{1}{\sigma_0^2}, -\frac{1}{\sigma_1^2}\right)\begin{pmatrix} \tilde{\sigma}_0^2 - \sigma_0^2 \\ \tilde{\sigma}_1^2 - \sigma_1^2 \end{pmatrix} + o_p(\Delta) \\
=& \log(\sigma_0^2/\sigma_1^2) + U^\tau \begin{pmatrix} \tilde{\sigma}_0^2 - \sigma_0^2 \\ \tilde{\sigma}_1^2 - \sigma_1^2 \end{pmatrix} + o_p(\Delta),
\end{aligned}$$

where $\Delta = ||\tilde{\sigma}_0^2 - \sigma_0^2|| + ||\tilde{\sigma}_1^2 - \sigma_1^2||$. Therefore,

$$\sqrt{n}\left[\log\left(\tilde{\sigma}_0^2/\tilde{\sigma}_1^2\right) - \log\left(\sigma_0^2/\sigma_1^2\right)\right] = U^\tau\sqrt{n}\begin{pmatrix} \tilde{\sigma}_0^2 - \sigma_0^2 \\ \tilde{\sigma}_1^2 - \sigma_1^2 \end{pmatrix} + o_p(\Delta)$$

The Theorem is completed after applying Theorem 3.1 to the above expression. □

Corollary 3.3. *As a result of Theorem 3.2, $\tilde{\sigma}_0^2/\tilde{\sigma}_1^2$ has a log-normal distribution. To be more specific, we have the following alternative asymptotic distribution of semiparametric estimator of the ratio of two variances.*

$$\tilde{\sigma}_0^2/\tilde{\sigma}_1^2 \xrightarrow{d} \mathcal{LN}\left[\log\left(\sigma_0^2/\sigma_1^2\right), \Omega_{semi}/n\right] \tag{18}$$

Both Theorem 3.2 and Corollary 3.3 can be used to define the test statistic for testing the following general hypothesis, for $r_0 > 0$,

$$H_0 : \sigma_0^2/\sigma_1^2 = r_0 \quad \text{versus} \quad H_a : \sigma_0^2/\sigma_1^2 \neq r_0. \tag{19}$$

The test statistic based on Theorem 3.1 is defined as

$$TS_1 = \frac{\sqrt{n}\left[\log(\tilde{\sigma}_0^2/\tilde{\sigma}_1^2) - \log r_0\right]}{\sqrt{\tilde{\Omega}_{semi}}}. \tag{20}$$

Under the null hypothesis $\sigma_0^2/\sigma_1^2 = r_0$, $TS_1 \xrightarrow{d} N(0,1)$. Note that $\tilde{\Omega}_{semi}$ is a consistent estimator of Ω that is explicitly given by

$$\tilde{\Omega}_{semi} = \tilde{U}^\tau \tilde{\Sigma}_{semi} \tilde{U} \tag{21}$$

where $\tilde{U} = (1/\tilde{\sigma}_0^2, -1/\tilde{\sigma}_1^2)^\tau$, $\tilde{\Sigma}_{semi} = \begin{pmatrix} \tilde{\Sigma}_{00}^s & \tilde{\Sigma}_{01}^s \\ \tilde{\Sigma}_{10}^s & \tilde{\Sigma}_{11}^s \end{pmatrix}$. The semiparametric estimators of the cell elements are given by

$$\begin{aligned}
\tilde{\Sigma}_{00}^s =&(1+\rho)\left\{\sum_{i=1}^n (T_i - \tilde{\mu}_0)^4 \tilde{p}_i - \left[\sum_{i=1}^n (T_i - \tilde{\mu}_0)^2 \tilde{p}_i\right]^2\right\} \\
&- \rho(1+\rho)\left\{\tilde{C}_2 - (\tilde{C}_0, \tilde{C}_1^\tau)A^{-1}\begin{pmatrix}\tilde{C}_0 \\ \tilde{C}_1\end{pmatrix}\right\}. \\
\tilde{\Sigma}_{11}^s =&\frac{1+\rho}{\rho}\left\{\sum_{i=1}^n (T_i - \tilde{\mu}_1)^4 \tilde{q}_i - \left[\sum_{i=1}^n (T_i - \tilde{\mu}_1)^2 \tilde{q}_i\right]^2\right\} \\
&- \frac{1+\rho}{\rho}\left\{\tilde{C}_2' - (\tilde{C}_0', \tilde{C}_1'^\tau)\tilde{A}^{-1}\begin{pmatrix}\tilde{C}_0' \\ \tilde{C}_1'\end{pmatrix}\right\}. \\
\tilde{\Sigma}_{01}^s =&[\Sigma_{10}^s]^\tau = (1+\rho)\left\{\tilde{C}_2'' - (\tilde{C}_0, \tilde{C}_1^\tau)\tilde{A}^{-1}\begin{pmatrix}\tilde{C}_0' \\ \tilde{C}_1'\end{pmatrix}\right\}.
\end{aligned} \tag{22}$$

Testing the hypothesis that two independent population variances are equal is equivalent to $r_0 = 1$ in hypothesis (19) and the test statistic (20).

4. Nonparametric Variance Ratio and Asymptotic Results

We consider the estimators of the population variances based on the two independent samples defined in (11). The following lemma is needed to study the asymptotic results of the non-parametric variance ratio.

Lemma 4.1. *Let $\{x_1, x_2, \cdots, x_n\}$ be a simple random sample taken from a population with cumulative distribution $G(x)$ and $\bar{x}$ be the sample mean. The sample variance $\hat{\sigma}^2$ satisfies*

$$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \xrightarrow{d} \mathcal{N}(0, \Theta), \tag{23}$$

where the variance of $\sqrt{n}\hat{\sigma}^2$ is given by

$$\Theta = \int_{-\infty}^{\infty} (x-\mu)^4 dG(x) - \left[\int_{-\infty}^{\infty} (x-\mu)^2 dG(x)\right]^2.$$

and μ is the population mean.

Proof. Let μ and σ^2 be the mean and variance of the population. Let $\bar{x}$ and $\hat{\sigma}^2$ be the estimators of the true mean and variance of the population. Note that

$$n\hat{\sigma}^2 = \sum_{i=1}^{n} (x_i - \mu)^2 - n(\bar{x} - \mu)^2.$$

We write

$$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (x_i - \mu)^2 - \sqrt{n}\sigma^2 - \sqrt{n}(\bar{x} - \mu)^2$$

$$= \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} (x_i - \mu)^2 - \sigma^2\right] - \sqrt{n}(\bar{x} - \mu)^2. \tag{24}$$

We next point out the following fact about the terms in (24).

$$\sqrt{n}(\bar{x} - \mu)^2 \xrightarrow{p} 0.$$

Note also that

$$E_G(X_i - \mu)^2 = \sigma^2 \quad \text{and} \quad \text{var}\big[(X_i - \mu)^2\big] = E_G(X_i - \mu)^4 - [E_G(X_i - \mu)^2]^2 = \mu_4 - \sigma^4,$$

where μ_4 is the 4th central moments of X and σ^4 is the square of the variance of X. By the Central Limit Theorem,

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} (X_i - \mu)^2 - \sigma^2\right] \xrightarrow{d} \mathcal{N}(0, \mu_4 - \sigma^4).$$

This implies that

$$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \xrightarrow{d} \mathcal{N}(0, \mu_4 - \sigma^4).$$

The proof is completed. □

Now we consider the two independent samples that satisfy model (1). Let $\bar{x}$ and $\hat{\sigma}_0^2$ be the sample mean and the sample variance of $\{x_1, x_2. \cdots, x_{n_0}\}$ taken from the population with distribution function $G_0(x)$. $\bar{y}$ and $\hat{\sigma}_1^2$ be the sample mean and the sample variance of $\{y_1, y_2. \cdots, y_{n_1}\}$ taken from the population with distribution function $G_1(y)$. Assume that the two samples are independent, then we have the following asymptotic distribution.

Theorem 4.2. *Let n_0 and n_1 be the sizes of the two samples specified in model (1) and $n = n_0 + n_1$. Let $\hat{\sigma}_0^2$ and $\hat{\sigma}_1^2$ be the nonparametric estimators of σ_0^2 and σ_1^2 respectively. Assume $\rho = n_1/n_0$ approaches a finite number as $n \to \infty$. We have*

$$\sqrt{n}\begin{pmatrix}\hat{\sigma}_0^2 - \sigma_0^2 \\ \hat{\sigma}_1^2 - \sigma_1^2\end{pmatrix} \xrightarrow{d} \mathcal{N}\left(0, \Sigma_{non}\right), \tag{25}$$

where

$$\Sigma_{non} = \begin{pmatrix}\Sigma_{00}^n & 0 \\ 0 & \Sigma_{11}^n\end{pmatrix}$$

and

$$\begin{aligned}\Sigma_{00}^n &= (1+\rho)\left\{\int_{-\infty}^{\infty}(u-\mu_0)^4 dG_0(u) - \left[\int_{-\infty}^{\infty}(u-\mu_0)^2 dG_0(u)\right]^2\right\}, \\ \Sigma_{11}^n &= \frac{1+\rho}{\rho}\left\{\int_{-\infty}^{\infty}(u-\mu_1)^4 dG_1(u) - \left[\int_{-\infty}^{\infty}(u-\mu_1)^2 dG_1(u)\right]^2\right\}.\end{aligned}$$

Proof. The proof is straightforward after using the result of the above Lemma 4.1. Let $n = n_0 + n_1$. Since $\hat{\sigma}_0^2 - \sigma_0^2 \xrightarrow{d} N\left[0, \frac{1}{n_0}(\mu_4^0 - \sigma_0^4)\right]$ and $\hat{\sigma}_1^2 - \sigma_1^2 \xrightarrow{d} N\left[0, \frac{1}{n_1}(\mu_4^1 - \sigma_1^4)\right]$, we have $\sqrt{n}\left(\hat{\sigma}_0^2 - \sigma_0^2\right) \xrightarrow{d} N\left[0, (1+\rho)(\mu_4^0 - \sigma_0^4)\right]$ and $\sqrt{n}\left(\hat{\sigma}_1^2 - \sigma_1^2\right) \xrightarrow{d} N\left[0, \frac{1+\rho}{\rho}(\mu_4^1 - \sigma_1^4)\right]$. Note that $\hat{\sigma}_0^2$ and $\hat{\sigma}_1^2$ are independent. The result of the Theorem follows immediately after we rewrite the aforementioned univariate asymptotic normal distributions into the form of a bivariate normal distribution. □

The following Theorem concerns the asymptotic distribution of the ratio of the two sample variances

Theorem 4.3. *Let $U = (1/\sigma_0^2, -1/\sigma_1^2)^\tau$. Under the condition of Theorem 4.2, we have*

$$\sqrt{n}\left[\log\left(\hat{\sigma}_0^2/\hat{\sigma}_1^2\right) - \log\left(\sigma_0^2/\sigma_1^2\right)\right] \xrightarrow{d} \mathcal{N}\left(0, \Omega_{non}\right), \tag{26}$$

where $\Omega_{non} = U^\tau \Sigma_{non} U$. Σ_{non} is explicitly specified in Theorem 4.2.

Proof. We use Taylor expansion to expand $\log(\hat{\sigma}_0^2/\hat{\sigma}_1^2)$ at (σ_0^2, σ_1^2) and obtain

$$\log(\hat{\sigma}_0^2/\hat{\sigma}_1^2) = \log(\sigma_0^2/\sigma_1^2) + \frac{1}{\sigma_0^2}(\hat{\sigma}_0^2 - \sigma_0^2) - \frac{1}{\sigma_1^2}(\hat{\sigma}_1^2 - \sigma_1^2) + o_p(\Delta).$$

where $\Delta = ||\hat{\sigma}_0^2 - \sigma_0^2|| + ||\hat{\sigma}_1^2 - \sigma_1^2||$. Therefore

$$\sqrt{n}\left[\log(\hat{\sigma}_0^2/\hat{\sigma}_1^2) - \log(\sigma_0^2/\sigma_1^2)\right] = \left(\frac{1}{\sigma_0^2}, -\frac{1}{\sigma_1^2}\right)\sqrt{n}\begin{pmatrix}\hat{\sigma}_0^2 - \sigma_0^2\\ \hat{\sigma}_1^2 - \sigma_1^2\end{pmatrix} \xrightarrow{d} \mathcal{N}\left(0, \Omega_{non}\right).$$

which completes the proof of Theorem 4.3. □

Corollary 4.4. *As a result of Theorem 4.3, $\hat{\sigma}_0^2/\hat{\sigma}_1^2$ has a log-normal distribution. To be more specific, we have the following alternative asymptotic distribution of the semiparametric estimator of the variance ratio.*

$$\hat{\sigma}_0^2/\hat{\sigma}_1^2 \xrightarrow{d} \mathcal{LN}\left[\log(\sigma_0^2/\sigma_1^2), \Omega_{non}/n\right]. \tag{27}$$

Using the result of Theorem 4.3, we establish a nonparametric procedure for testing the hypothesis of (19). The test statistic is similarly defined by

$$TS_2 = \frac{\sqrt{n}\left[\log(\hat{\sigma}_0^2/\hat{\sigma}_1^2) - \log(r_0)\right]}{\sqrt{\Omega_{non}}}. \tag{28}$$

Under the null hypothesis $\sigma_0^2/\sigma_1^2 = r_0$, $TS_2 \xrightarrow{d} N(0,1)$. Note that Ω_{non} in TS_2 can be estimated using the central moments from the two sample data sets when in implementing the test.

5. Asymptotic Relative Efficiency of Parameter Estimators and Tests

We have introduced parametric and semiparametric tests for testing the equality of two independent population variances based on the variance ratio in early sections. In this section, we present large sample comparisons between the nonparametric and semiparametric estimators of the variance ratios and the nonparametric and semiparametric tests for the equality of variances as well. We use Pitman's asymptotic relative efficiency as a metric to compare the proposed large sample procedures.

Lemma 5.1. *Under the same notations in 3.1 and Theorem 4.3, we have*

$$var\left[\log\left(\frac{\tilde{\sigma}_0^2}{\tilde{\sigma}_1^2}\right)\right] < var\left[\log\left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)\right].$$

Proof. From the proof of Theorem 3.2 of Wan et al. Wan et al (2016), we have

$$\begin{aligned}\text{var}(\tilde{\sigma}_0^2 - \hat{\sigma}_0^2) =& \frac{\rho}{n_0}\left\{C_2 - (C_0, C_1^\tau)A^{-1}\begin{pmatrix}C_0\\ C_1\end{pmatrix}\right\}.\\ \text{var}(\tilde{\sigma}_1^2 - \hat{\sigma}_1^2) =& \frac{1}{n_1}\left\{C_2' - (C_0', C_1'^\tau)A^{-1}\begin{pmatrix}C_0'\\ C_1'\end{pmatrix}\right\}.\\ \text{cov}(\tilde{\sigma}_0^2 - \hat{\sigma}_0^2, \tilde{\sigma}_1^2 - \hat{\sigma}_1^2) =& -\frac{1}{n_0}\left\{C_2'' - (C_0, C_1^\tau)A^{-1}\begin{pmatrix}C_0'\\ C_1'\end{pmatrix}\right\}.\end{aligned} \tag{29}$$

After performing some straightforward calculations, we have

$$\text{var}\left[\log\left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)\right] - \text{var}\left[\log\left(\frac{\tilde{\sigma}_0^2}{\tilde{\sigma}_1^2}\right)\right] = \frac{1}{n}U^\tau\left(\Sigma_{\text{non}} - \Sigma_{\text{semi}}\right)U$$

$$= U^\tau \begin{pmatrix} \text{var}(\tilde{\sigma}_0^2 - \hat{\sigma}_0^2) & \text{cov}(\tilde{\sigma}_0^2 - \hat{\sigma}_0^2, \tilde{\sigma}_1^2 - \hat{\sigma}_1^2) \\ \text{cov}(\tilde{\sigma}_0^2 - \hat{\sigma}_0^2, \tilde{\sigma}_1^2 - \hat{\sigma}_1^2) & \text{var}(\tilde{\sigma}_1^2 - \hat{\sigma}_1^2) \end{pmatrix} U$$

$$= \frac{1}{\sigma_0^4}\text{var}(\tilde{\sigma}_0^2 - \hat{\sigma}_0^2) - \frac{2}{\sigma_0^2\sigma_1^2}\text{cov}(\tilde{\sigma}_0^2 - \hat{\sigma}_0^2, \tilde{\sigma}_1^2 - \hat{\sigma}_1^2) + \frac{1}{\sigma_1^4}\text{var}(\tilde{\sigma}_1^2 - \hat{\sigma}_1^2)$$

$$= \text{var}\left(\frac{\tilde{\sigma}_0^2 - \hat{\sigma}_0^2}{\sigma_0^2} - \frac{\tilde{\sigma}_1^2 - \hat{\sigma}_1^2}{\sigma_1^2}\right) > 0,$$

which completes the proof. □

We will discuss the asymptotic relative efficiency respectively in the following subsections.

5.1. Asymptotic Relative Efficiency of Estimators of Variance Ratio

Since both nonparametric and semiparametric estimators of the variance ratio are asymptotically unbiased, we assess the performance of the estimators using asymptotic relative efficiency which is defined as the ratio of the two corresponding variances.

Theorem 5.2. *The empirical likelihood-based semiparametric estimator of the variance ratio is asymptotically more efficient than its non-parametric counterpart. That is,*

$$var\left(\frac{\tilde{\sigma}_0^2}{\tilde{\sigma}_1^2}\right) < var\left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right). \tag{30}$$

Proof. Let $V = (1/\sigma_1^2, -\sigma_0^2/\sigma_1^4) = (\sigma_0^2/\sigma_1^2)U^\tau$. U is specified in Theorem 4.3. We take first-order Taylor expansion on the two estimators of the variance ratio at the true values of the two population variances σ_0^2 and σ_1^2 as follows.

$$\frac{\tilde{\sigma}_0^2}{\tilde{\sigma}_1^2} = \frac{\sigma_0^2}{\sigma_1^2} + V^\tau\begin{pmatrix}\tilde{\sigma}_0^2 - \sigma_0^2 \\ \tilde{\sigma}_1^2 - \sigma_1^2\end{pmatrix} + o_p(\Delta) \tag{31a}$$

$$\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} = \frac{\sigma_0^2}{\sigma_1^2} + V^\tau\begin{pmatrix}\hat{\sigma}_0^2 - \sigma_0^2 \\ \hat{\sigma}_1^2 - \sigma_1^2\end{pmatrix} + o_p(\Delta) \tag{31b}$$

Taking the variance of (31b) and (31a) and then taking the difference of the two equations yield

$$\operatorname{var}\left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right) - \operatorname{var}\left(\frac{\tilde{\sigma}_0^2}{\tilde{\sigma}_1^2}\right) = \frac{1}{n} V^\tau \left(\Sigma_{non} - \Sigma_{semi}\right) V$$

using the same notations and similar steps of simplification in the proof of Lemma 5.1, we have

$$\frac{1}{n} V^\tau \left(\Sigma_{non} - \Sigma_{semi}\right) V = \frac{\sigma_0^4}{\sigma_1^4} \times U^\tau \begin{pmatrix} \operatorname{var}(\tilde{\sigma}_0^2 - \hat{\sigma}_0^2) & \operatorname{cov}(\tilde{\sigma}_0^2 - \hat{\sigma}_0^2, \tilde{\sigma}_1^2 - \hat{\sigma}_1^2) \\ \operatorname{cov}(\tilde{\sigma}_0^2 - \hat{\sigma}_0^2, \tilde{\sigma}_1^2 - \hat{\sigma}_1^2) & \operatorname{var}(\tilde{\sigma}_1^2 - \hat{\sigma}_1^2) \end{pmatrix} U$$

$$= \frac{\sigma_0^4}{\sigma_1^4} \operatorname{var}\left(\frac{\tilde{\sigma}_0^2 - \hat{\sigma}_0^2}{\sigma_0^2} - \frac{\tilde{\sigma}_1^2 - \hat{\sigma}_1^2}{\sigma_1^2}\right) > 0.$$

This means that the semiparametric variance ratio is asymptotically more efficient than the non-parametric estimate of the variance ratio. □

Next, we discuss the asymptotic relative efficiency of the two proposed large sample tests based on the variance ratio.

5.2. Asymptotic Relative Efficiency of Tests

Since both non-parametric (28) and semi-parametric (20) tests are consistent, the asymptotic power under the global alternatives approaches unity. To assess the performance of these tests, we compare the sensitivity of these tests under the local alternatives that converge to the null hypothesis. The Pitman's asymptotic relative efficiency (ARE) will be used as a measure for asymptotic comparison between tests (28) and (20). The idea behind Pitman's relative efficiency between two tests is to compare the sample sizes required to achieve the same power at a given common nominal level α. The smaller the sample size, the better the corresponding test.

For more detail on Pitman's efficiency and other efficiency metrics, we refer to monographs of Serfling (1980) and van der Vaart (2000). In this section, without loss of generality, we consider the following right-tailed test.

$$H_0: \ \frac{\sigma_0^2}{\sigma_1^2} \le 1 \quad \text{and} \quad H_a: \ \frac{\sigma_0^2}{\sigma_1^2} > 1 \tag{32}$$

The local alternatives in this section will be chosen in the following form, for some $h > 0$,

$$H_n: \quad r_n = \log\left[\left(\sigma_0^2/\sigma_1^2\right)\left(1 + 1/\sqrt{n}\right)^h\right] = \log\left[\frac{\sigma_0^2(1 + 1/\sqrt{n})^h}{\sigma_1^2}\right] \tag{33}$$

Apparently, $\lim_{n\to\infty} r_n = r_0 = \sigma_0^2/\sigma_1^2$. Next, we derive the asymptotic distribution of $\log\left[\tilde{\sigma}_0^2/\tilde{\sigma}_1^2\right]$ under local alternatives H_n for $i = 1, 2, \cdots,$.

The regularity conditions of Pitman's approach are based on the sampling distribution of the test statistic under alternatives. For the semiparametric test semiparametric (20), we expand the semiparametric log variance ratio, denoted by $\tilde{T}_n = \log(\tilde{\sigma}_0^2/\tilde{\sigma}_1^2)$, under the local alternatives at $(\sigma_0^2(1+1/\sqrt{n})^h, \sigma_1^2)$ to find the sampling distribution and obtain

$$\tilde{T}_n = \log\left[\frac{\tilde{\sigma}_0^2}{\tilde{\sigma}_1^2}\right] = \log\left[\frac{\sigma_0^2(1+1/\sqrt{n})^h}{\sigma_1^2}\right] + U^\tau(n)\begin{pmatrix}\tilde{\sigma}_0^2 - \sigma_0^2(1+1/\sqrt{n})^h \\ \tilde{\sigma}_1^2 - \sigma_1^2\end{pmatrix} + o_p(\Delta(n)) \tag{34}$$

where

$$U(n) = \left(\frac{1}{\sigma_0^2(1+1/\sqrt{n})^h}, -\frac{1}{\sigma_1^2}\right)^\tau \quad \text{and} \quad \tilde{\Delta}(n) = ||\tilde{\sigma}_0^2 - \sigma_0^2(1+1/\sqrt{n})^h|| + ||\tilde{\sigma}_1^2 - \sigma_1^2|| \tag{35}$$

Furthermore, we denote

$$\text{var}_{H_n}(\tilde{T}_n) = U^\tau(n)\Sigma_{semi}(n)U(n) = \Omega_{semi}(n) \tag{36}$$

Next we list several regularity conditions based on the limiting behaviors of $\tilde{T}_n$.

(C1). Asymptotic normality of $\tilde{T}_n$ under H_n:

$$\frac{\sqrt{n}(\tilde{T}_n - E_{H_n}[\tilde{T}_n])}{\sqrt{\Omega_{semi}(n)}} \xrightarrow{H_n} N(0,1). \tag{37}$$

From (34), we have

$$E_{H_n}[\tilde{T}_n] = \log\left[\frac{\sigma_0^2(1+1/\sqrt{n})^h}{\sigma_1^2}\right]$$

(C2).

$$\lim_{n\to\infty}\Omega(n) = \lim_{n\to\infty}[U^\tau(n)\Sigma_{semi}U(n)] = U^\tau\Sigma_{semi}U = \Omega_{semi}.$$

(C3). As $n \to \infty$,

$$\lim_{n\to\infty}\left[\frac{E_{H_n}(\tilde{\sigma}_0^2/\tilde{\sigma}_1^2) - E_{H_0}(\tilde{\sigma}_0^2/\tilde{\sigma}_1^2)}{\sqrt{\text{var}_{H_n}(\tilde{\sigma}_0^2/\tilde{\sigma}_1^2)}}\right] = \lim_{n\to\infty}\frac{h\log(1+1/\sqrt{n})}{\sqrt{\Omega_{semi}}/\sqrt{n}} = \frac{h}{\sqrt{\Omega_{semi}}} \tag{38}$$

The limit in (38) is customarily called the efficacy of the semiparametric test (20). With the above regularity conditions C1-C3, we can calculate the power function of test (20) based on local alternatives in the following theorem.

Theorem 5.3. *Under the regularity conditions C1-C3 and given nominal level α, the asymptotic limiting power of test (20) under the local alternative (33) is given by*

$$\lim_{n\to\infty}\pi_n^S(H_n:n) = 1 - \Phi\left(z_\alpha - \frac{h}{\sqrt{\Omega_{semi}}}\right) \tag{39}$$

where z_α is the value that satisfies $\Phi(z_\alpha) = 1-\alpha$, $\Phi()$ is the cumulative distribution function of the standard normal distribution.

Proof. Note that, for a given nominal level α, $\pi^S(H_0 : n) = \alpha = P(\tilde{T}_n > t^S_{\alpha,n}|H_0)$. This means

$$\frac{t^S_{\alpha,n} - \log(\sigma_0^2/\sigma_1^2)}{\sqrt{\Omega_{semi}/n}} = z_\alpha,$$

that is equivalent to

$$t^S_{\alpha,n} = \log(\sigma_0^2/\sigma_1^2) + z_\alpha\sqrt{\Omega_{semi}/n} = E_{H_0}(\tilde{T}_n) + z_\alpha\sqrt{\Omega_{semi}/n}. \tag{40}$$

The power function is defined by

$$\pi^S(H_n : n) = P(\tilde{T}_n > t^S_{\alpha,n}|H_n) = P\left(\frac{\tilde{T}_n - E_{H_n}(\tilde{T}_n)}{\sqrt{\Omega_{semi}(n)/n}} > \frac{t^S_{\alpha,n} - E_{H_n}(\tilde{T}_n)}{\sqrt{\Omega_{semi}(n)/n}}\right) \tag{41}$$

Let

$$z_n = \frac{\tilde{T}_n - E_{H_n}(\tilde{T}_n)}{\sqrt{\Omega_{semi}(n)/n}}$$

Then from (C1), $z_n \to N(0,1)$. Therefore,

$$\pi^S(H_n : n) = 1 - \Phi\left(\frac{t^S_{\alpha,n} - E_{H_n}(\tilde{T}_n)}{\sqrt{\Omega_{semi}(n)/n}}\right)$$

Note that, using (40), we obtain

$$\frac{t^S_{\alpha,n} - E_{H_n}(\tilde{T}_n)}{\sqrt{\Omega_{semi}(n)/n}} = \frac{E_{H_0}(\tilde{T}_n) + z_\alpha\sqrt{\Omega_{semi}/n} - E_{Hn}(T_n)}{\sqrt{\Omega_{semi}(n)/n}}$$

$$= z_\alpha - \frac{E_{H_0}(\tilde{T}_n) - E_{H_n}(\tilde{T}_n)}{\sqrt{\Omega_{semi}(n)/n}} = z_\alpha - \frac{h\log(1 + 1/\sqrt{n})}{\sqrt{\Omega_{semi}(n)}/\sqrt{n}}. \tag{42}$$

The limit of the last term in the above equation is given below by using (C2) and (C3).

$$\lim_{n\to\infty} \frac{h\log(1 + 1/\sqrt{n})}{\sqrt{\Omega_{semi}(n)}/\sqrt{n}} = \frac{h}{\sqrt{\Omega_{semi}}}.$$

Using the Continuous Mapping Theorem, we obtain the following limiting power function under local alternatives.

$$\lim_{n\to\infty} \pi^S_n(H_n : n) = 1 - \Phi\left(z_\alpha - \frac{h}{\sqrt{\Omega_{semi}}}\right). \tag{43}$$

This completes the proof. □

We can also establish a Theorem analogous to Theorem 5.3 based on nonparametric test (28) as follows for some $h' > 0$.

Theorem 5.4. *Under similar regularity conditions, we have the following limiting power function for non-parametric test (28) in the following*

$$\lim_{n\to\infty} \pi_n^N(H_n : n) = 1 - \Phi\left(z_\alpha - \frac{h'}{\sqrt{\Omega_{non}}}\right), \tag{44}$$

where $h' > 0$.

We now look at the sample sizes required for both semiparametric test (20) and non-parametric test (28) to attain the power with the common nominal level α. The two limiting powers are equal if and only if

$$1 - \Phi\left(z_\alpha - \frac{h}{\sqrt{\Omega_{semi}}}\right) = 1 - \Phi\left(z_\alpha - \frac{h'}{\sqrt{\Omega_{non}}}\right)$$

which implies

$$\frac{h}{\sqrt{\Omega_{semi}}} = \frac{h'}{\sqrt{\Omega_{non}}}, \quad \text{equivalently,} \quad \frac{h}{h'} = \frac{\sqrt{\Omega_{semi}}}{\sqrt{\Omega_{non}}} \tag{45}$$

Since both power functions are calculated using the local alternatives, this means

$$\frac{h}{\sqrt{n}} = \frac{h'}{\sqrt{n'}} \tag{46}$$

where n and n' are corresponding sample sizes of semiparametric (20) and non-parametric tests (28). From (45) and (46), we obtain

$$\frac{n}{n'} = \frac{\Omega_{semi}}{\Omega_{non}} < 1. \tag{47}$$

The last inequality is based on the result of Lemma 5.1. We summarize the above result in the following main Theorem of this section.

Theorem 5.5. *Under model (1), empirical likelihood-based semiparametric test (20) is asymptotically more efficient than the non-parametric test (28).*

6. Bootstrap Semiparametric Test

We have discussed the asymptotic properties of the semi-parametric estimator of the variance ratio and the hypothesis testing on the equality of two variances. In this section, we present a model-based semi-parametric Bootstrap test for the equality of two independent population variances. The Bootstrap sampling plan uses the semi-parametric estimators of the two population distributions specified in (8). The Bootstrap test will be based on the confidence interval approach as suggested by Hall and Wilson (1991).

The Bootstrap test algorithm is outlined in the following:

Bootstrap Algorithm

Let $\{x_1, x_2, \cdots, x_{n_0}\}$ and $\{y_1, y_2, \cdots, y_{n_1}\}$ be the original samples taken from two populations $G_0(x)$ and $G_1(y)$. Let the pooled sample $T = \{T_1, T_2, \cdots, T_n\} = \{x_1, x_2, \cdots, x_{n_0}, y_1, y_2, \cdots, y_{n_1}\}$. Under model (1), the semiparametric estimators of $G_0(t)$ and $G_1(t)$, denoted by $\tilde{G}_0(t)$ and $\tilde{G}_1(t)$, are given in (8). Let $\tilde{w} = \exp(\tilde{\alpha}+\tilde{\beta}^\tau r(x))$. We write $\tilde{p}_i = d\tilde{G}_0(T_i)$ and $\tilde{q}_i = \tilde{p}_i\tilde{w}_i = d\tilde{G}_1(T_i)$.

Step 1. Fit model (1) to the pooled data and estimate $\tilde{G}_0(t)$ and $\tilde{G}_1(t)$, and obtain $\tilde{p}_i$ and $\tilde{q}_i$ for $i = 1, 2, \cdots, n$, as explicitly in (6) and (7).
Step 2. Take a weighted bootstrap sample of size n_0 with weights $\tilde{p}_i$ from the pooled data set, denoted by X^{*b}, and another bootstrap sample of size n_1 with weights $\tilde{q}_i$, denoted by Y^{*b}.
Step 3. Fit model (1) to the pooled data $T^{*b} = \{X^{*b} \bigcup Y^{*b}\}$ and calculate the bootstrap variance $\tilde{\sigma}_1^{2*}$ and $\tilde{\sigma}_2^{2*}$ as defined in (9). Define $\theta^{*b} = \tilde{\sigma}_1^{2*}/\tilde{\sigma}_2^{2*}$.
Step 4. Repeat *S*tep 2 and *S*tep 3 B times and obtain B semi-parametric variance ratios $\{\theta^{*1}, \theta^{*2}, \cdots, \theta^{*B}\}$.
Step 5. $100(1-\alpha)\%$ two-sided bootstrap percentile confidence interval of the variance ratio θ is defined by $(\theta^*_{\alpha/2}, \theta^*_{1-\alpha/2})$. If the claimed variance ratio under H_0 falls outside the Bootstrap percentile confidence interval, we reject the null hypothesis H_0 at the level α.

7. Power Analysis via Simulation Study

We have developed the asymptotic theory of the proposed two-sample semi-parametric tests of population variances. In this section, we conduct numerical experiments to study the small sample properties of the tests.

To perform power comparisons between the proposed tests and some existing tests, We use the well-known normal and Gamma distributions in the simulation studies.

The hypothesis test in the study is

$$H_0 : \sigma_1^2/\sigma_2^2 = 1 \quad \text{v.s.} \quad H_a : \sigma_1^2/\sigma_2^2 \neq 1$$

The tests to be included in the simulation studies are parametric F test (F variance ratio, FVR), non-parametric variance ratio (NPVR) test (Section 4), semi-parametric variance ratio (SPVR) test (Section 3), semi-paramteric variance difference (SPVD) test in Wan et al (2016) and the bootstrap semi-parametric variance ratio test (BSPVR) proposed in the previous section.

For the four non-bootstrap tests, we simulate 1000 Monte Carlo samples from the theoretical populations that satisfy the assumption of model (1) and perform each of the four tests based on different parameter configurations so the true variance ratio (or difference) differs from the one claimed in the null hypothesis.

For the bootstrap semi-parametric variance ratio test, for each of the 1000 simulations, we generate 1000 bootstrap samples as described in the previous section to find the bootstrap percentile confidence interval and use the two-sided confidence interval to test the hypothesis on variance ratio θ.

The nominal significance level 0.05 will be used in the following simulation studies.

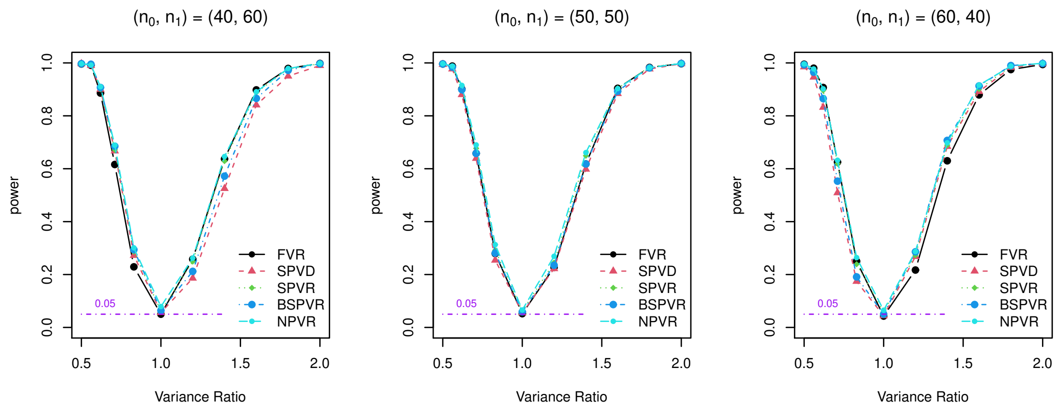

Figure 1. Power curves of various tests based on the samples taken from two independent normal populations with different sample sizes.

7.1. Case 1: Normal Population

We first consider the case that both samples are taken from normal populations $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$. Under model (1) with $r(x) = (x, x^2)^\tau$ and $\beta = (\beta_1, \beta_2)^\tau$. Then the three regression coefficients of model (1) are specified in the following.

$$\alpha = \log \frac{\sigma_1}{\sigma_2} + \frac{1}{2}\left(\frac{\mu_1^2}{\sigma_1^2} - \frac{\mu_2^2}{\sigma_2^2}\right) \quad , \quad \beta_1 = \frac{\mu_2}{\sigma_2^2} - \frac{\mu_1}{\sigma_1^2}, \quad \text{and} \quad \beta_2 = \frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)$$

We consider sample sizes of $(n_0, n_1) = \{(40, 60), (50, 50), (60, 40)\}$. Also, we let $\mu_1 = 0, \mu_2 = 1, \sigma_1 = (5/10, 5/9, 5/8, 5/7, 5/6, 1, 6/5, 7/5, 8/5, 9/5, 10/5)$ and $\sigma_2 = 1$. For each combination of $(\mu_1, \mu_2, \sigma_1, \sigma_2, n_0, n_1)$, we generate 1000 independent sets of combined Monte Carlo samples from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$. The power curves of all tests are given in Figure 1.

As anticipated, we can see from Figure 1 that all tests behave similarly when both samples are taken from normal populations.

7.2. Case 2: Gamma Population

Let $G_0(x)$ and $G_1(x)$ be Gamma(α_1, λ_1) and Gamma(α_2, λ_2) with density functions

$$g_0(x) = \frac{\lambda_1^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} e^{-\lambda_1 x} \quad \text{and} \quad g_1(x) = \frac{\lambda_2^{\alpha_2}}{\Gamma(\alpha_2)} x^{\alpha_2 - 1} e^{-\lambda_2 x}, \tag{48}$$

respectively. Under model (1) with

$$r(x) = (x, \log(x))^\tau \quad \text{and} \quad \beta = (\beta_1, \beta_2)^\tau.$$

Similarly, we can specify the three regression coefficients of (1) with the given Gamma parameters in the following.

$$\alpha = \log\left(\frac{\Gamma(\alpha_1)\lambda_2^{\alpha_2}}{\Gamma(\alpha_2)\lambda_1^{\alpha_1}}\right), \beta_1 = \lambda_1 - \lambda_2 \quad \text{and} \quad \beta_2 = \alpha_2 - \alpha_1$$

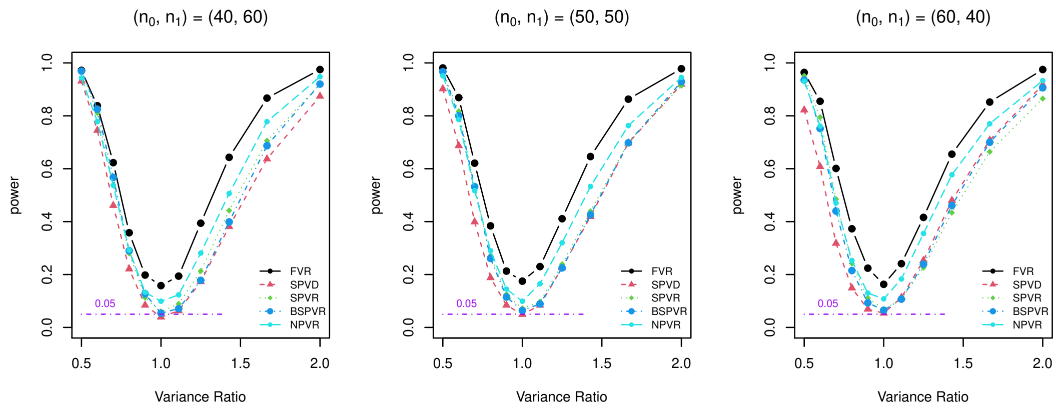

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Figure 2. Power curves of various tests based on the samples taken from two independent Gamma populations with different sample sizes.

For given $\alpha_1 = 4, \alpha_2 = 2, \lambda_1 = \sqrt{2}$, we find the true ratio of the two variances to be

$$\theta = \frac{\alpha_1}{\lambda_1^2} \times \frac{\lambda_2^2}{\alpha_2} = \frac{4}{2} \times \frac{\lambda_2^2}{2} = \lambda_2^2$$

We now consider sample sizes of $(n_0, n_1) = (40, 60), (50, 50), (60, 40)$ and $\lambda_2 = (5/10, 6/10, 7/10, 8/10, 9/10, 1, 10/9, 10/8, 10/7, 10/6, 10/5)$. The choice of λ_2 results different variance ratio $(0.25, 0.36, 0.49, 0.64, 0.81, 1.00, 1.23, 1.56, 2.04, 2.78, 4.00)$. For each combination of $(\alpha_1, \alpha_2, \lambda_1, \lambda_2, n_0, n_1)$, we generate 1000 independent sets of combined Monte Carlo samples from Gamma(α_1, λ_1) and Gamma(α_2, λ_2) and fit the density ratio model (1) to each of the pooled data and then perform all five tests. The simulated power of all tests is summarized in Figure 2.

From the power curves in Figure 2, we can see the following patterns

(1) The parametric F test (PVR) produces was inappropriate since both populations are not normally distributed. This also reflected in all three curves since the observed p-value and actual significance of the parametric F test (FVR) are significantly different.

(2) The non-parametric variance ratio (NPVR) test described in Section 4 also has a minor issue of power inflation since the observed p-value is slightly bigger than the actual significance level.

(3) The semiparametric variance ratio (SPVR) and the bootstrap semi-parametric variance ratio (BSPVR) are marginally more powerful than the semi-parametric variance difference (SPVD) test.

7.3. Exploring Impact of Sample Size Allocation

For a given overall sample size, the optimal allocation of sub-sample sizes to achieve the best statistical power is of practical importance in problems of multiple comparisons. Numerous studies about this topic related to comparing population means have been conducted in the statistics literature and relevant disciplines as well under various assumptions about the underlying populations. See, for example, the recent work of

Nam (2009), Happ et. al. (2019), Jan and Shieh (2011), and Rusticus and Lovato (2014), etc. Rusticus and Lovato (2014) conducted a comprehensive simulation study about factors that impact the power of four different tests for two- and three-sample comparisons using the same population variance and concluded that equal sample sizes are more powerful than unequal sample sizes and the power decreases as samples become increasingly imbalanced.

The proposed semiparametric tests are model-based procedures. The underlying special density ratio model (1) is a retrospective logistic regression model. Since the empirical-likelihood-based semiparametric estimators of the regression coefficients and the estimated semiparametric cell probabilities (6) are functions of the ratio of the two sample sizes ρ. Hence, the variance of the proposed test statistic is also a function of ρ. This means that for given overall sample size in the proposed test, the sample size ratio impacts the variance of the test statistic. Therefore, the sample size ratio impacts the power of the proposed tests.

To see how the ratio of the two sample sizes impacts the power of the tests, we simulate samples from both normal and Gamma populations with different parameter configurations (see the caption of Figure 3) to meet the assumptions of the underlying model (1). The overall sample size in this simulation study is 200. The sub-sample sizes sample sizes are chosen to be $n_0 = (20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180)$ and $n_1 = (180, 170, 160, 150, 140, 130, 120, 110, 100, 90, 80, 70, 60, 50, 40, 30, 20)$. The power curves were drawn based on the different ratios of sample sizes, $n_0/n_1 = c(1/9, 3/17, 1/4, 1/3, 3/7, 7/13, 1/3, 9/11, 1, 11/9, 3/2, 13/7, 4, 3, 4, 17/3, 9)$.

For normal populations, we simulate two scenarios: (1). $N_{01}(0,(5/7)^2)$ and $N_1(1,1)$ with a fixed variance ratio $VR = 25/49 < 1$; (2). $N_{01}(0,(7/5)^2)$ and $N_1(1,1)$ with a fixed variance ratio $VR = 49/25 > 1$. For Gamma populations, We also simulate two scenarios: (1). $\text{Gamma}_{01}(4, \sqrt{2})$ and $\text{Gamma}_1(2, 3/5)$ with a fixed variance ratio $VR = 9/50 > 1$; (2). $\text{Gamma}_{02}(4, \sqrt{2})$ and $\text{Gamma}_1(2, 5/3)$ with a fixed variance ratio $VR = 18/25 < 1$.

With the above given overall sample size of 200 and fixed variance ratios (different alternatives), we simulate sub-samples with the preset ratio of sample sizes and calculate estimated power. The results are summarized in Figure 3.

We can observe the following patterns regarding the sample ratio and the power of the proposed two-sample variance tests from Figure 3.

1. If both underlying populations in the study are from normal and gamma families, the power of both SPVR and BSPVR tests decreases as the sample size ratio deviates far from unity when the variance ratio is either greater than or less than 1.

2. For the fixed overall sample size of 200 and variance ratios of the two underlying populations from either normal and Gamma families, both proposed tests achieve the best power when both sub-sample sizes are approximately the same. This basically says that the balanced design will result in a better statistical power.

Remark 1. Although the underlying populations in the simulation were chosen from normal and gamma families respectively, the actual underlying populations are arbitrary provided that model (1) holds. The above observations are dependent on model (1) but not particular distributions.

Remark 2. The optimal sub-sample allocation in the two-sample nonparametric variance test discussed in Section 4 is not approximately evenly split the fixed overall sample size. For example, Let $F_0(x)$ and $F_1(x)$ be the distributions of the two corre-

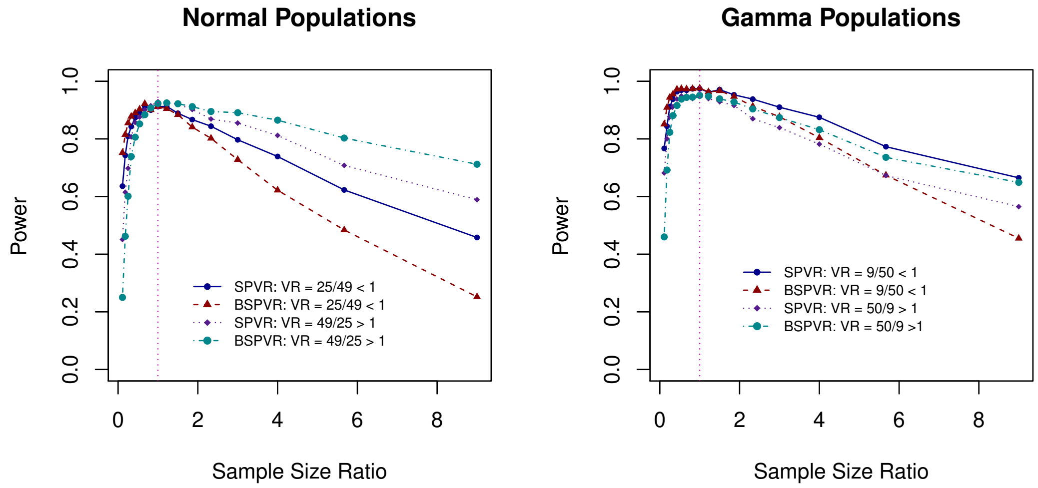

Figure 3. Power curves of semiparametric and bootstrap semiparametric tests of equal variance based on the samples taken from two independent populations with different sample sizes. *Left panel:* The two normal distributions are: $[N_{01}(0, 5^2/7^2), N_1(1, 1)]$ and $[N_{02}(0, 7^2/5^2), N_1(1, 1)]$. *Right panel:* $[Gamma_0(4, \sqrt{2}), Gamma_{11}(2, 3/5)]$ and $[Gamma_0(4, \sqrt{2}), Gamma_{12}(2, 5/3)]$

sponding underlying populations, we can see from Theorem 4.2 and Theorem 4.3 that the variance of the test statistic is dependent on both the sample size ratio, ρ and the ratio of functions of moments associated with corresponding unrelated populations $F_0(x)$ and $F_1(x)$ as well.

Remark 3. Model (1) assumes the relationship between the two density functions of the underlying populations to be $g_1(x) = \exp(\alpha + \beta^\tau x)g_0(x)$. This means one density is uniquely determined by the other. Hence, the variance of the proposed semiparametric test is a function of only the sample size ratio (ρ) and the baseline distribution $G_0(x)$. This implies that the power function of the test is only a function of the sample size ratio for given overall sample size and one of the populations.

8. Numerical Examples

In this section, we present two numerical examples that were analyzed by various authors in the past. The purpose of presenting thes numerical examples is to demonstrate how to implement the proposed tests. The non-parametric test (NPVR) and the semiparametric test of the difference of variances (SPVD) by Wan et al (2016) were also used baseline tests.

Example 1. The data used in this example come from a case-control study conducted in Mayo Clinic in which sera from $n_0 = 51$ 'control' patients with pancreatitis and $n_1 = 90$ 'cases' with pancreatic cancer were studied with a cancer antigen (CA125) and with a carbohydrate antigen (CA19-9). Wiend et al (1989) used this data in analyzing sensitivity-specificity for comparing two diagnostic markers. Since the original data are of several orders of magnitude and Shapiro-Wilk's normality test yields p-values close to 0, a log transformation is used in our data analysis. However, the transformed data still could not pass the Shapiro-Wilk's normality test at $\alpha = 0.05$, the usual parametric F test cannot be used. Qin and Zhang (2003) performed a Kolmogorov-Smirnov like goodness-of-fit test of model (1) with $r(x) = (x, x^2)^\tau$ that was fitted to this data. The model has a good fit for the data. We use the sample under the model (1) and obtained the corresponding $\tilde{p}_i$ and $\tilde{q}_i$ for $i = 1, 2, \cdots, n$.

The semiparametric likelihood estimate of parameters are given by $(\tilde{\alpha}, \tilde{\beta}_1, \tilde{\beta}_2) = (0.560, -1.914, 0.451)$. We perform the four tests used in the simulation study and the report the statistics in the following table 1

Table 1. Summarized statistics of Example 1.

	Name of Tests			
	SPVD	SPVR	BSPVR (C.I.)	NPVR
TS	-7.841	-8.940	(0.080, 0.209)	-9.445
p-value[a]	< 0.001	< 0.001	NA[b]	< 0.001

[a]Based on the asymptotic normal distribution of test statistics.
[b] The H_0 was rejected at level 0.05.

All tests reject the hypothesis $H_0 : \sigma_1^2/\sigma_2^2 = 1$ v.s. $H_1 : \sigma_1^2/\sigma_2^2 \neq 1$.

Example 2. The data set used in this example was collected from research by Reaven and Miller (1979) on a diabetes study. The steady-state plasma glucose (SSPG) levels based on the oral glucose tolerance test were taken from a subset of 109 subjects that were used in this example. Among these subjects, $n_0 = 76$ subjects were normal and $n_1 = 33$ subjects had overt diabetes. We are interested in testing the equality of the variances of SSPG levels of both diseased and disease-free populations. As anticipated, the Shapiro-Wilk test for normality fails the data. Wan et al (2016) fit model (1) with $r(x) = x$ to the data and performed a Kolmogorov-Smirnov test. The test result indicates that model (1) fits the data well.

We first fit the density ratio model (1) to the data and obtain $\tilde{\alpha} = -6.278$ and $\tilde{\beta} = 0.031$. Using the resulting $\tilde{p}_i$ and $\tilde{q}_i$ defined in (6) and (7), respectively, to estimate the weighted variances $\tilde{\sigma}_1^2$ and $\tilde{\sigma}_2^2$.

Next, we perform the four tests for the hypothesis:

$$H_0 : \sigma_1^2/\sigma_2^2 = 1 \quad \text{v.s.} \quad H_1 : \sigma_1^2/\sigma_2^2 \neq 1.$$

The statistics of the tests are summarized in the following Table 2

Table 2. Summarized statistics of Example 2.

	Name of Tests			
	SPVD	SPVR	BSPVR (C.I.)	NPVR
TS	-2.350	-2.804	(0.246, 0.820)	-3.052
p-value[a]	0.019	0.005	NA[b]	0.002

[a]Based on the asymptotic normal distribution of test statistics.
[b] The H_0 was rejected at level 0.05.

Similar to the results we obtained in Example 1, all four tests reject the null hypothesis of $\sigma_1^2/\sigma_2^2 = 1$.

Remark 4. The SPVD, SPVR and BSSPVR are same type of model-based test in which the underlying model allows information sharing in the two independent samples. The two-sample nonparametric test of variances (NPVR) does not use the shared information from both samples and is less powerful than the three semiparametric procedures. It is within our expectation that the resulting statistical decisions of the four tests are the same.

Remark 5. Since the proposed two-sample semiparametric test of variance is developed based on model (1), a goodness-of-fit test such as Qin and Zhang (1997) should be conducted before using this test as we did in the examples.

9. Conclusion

In this paper, we developed two semiparametric tests (SPVR, BSPVR) for testing the ratio of two population variances based on the assumption that the two population distributions have a structural relationship specified in the underlying model.

We use the nonparametric test (NPVR) of variance ratio as the baseline and study the relative efficiency of the proposed semiparametric estimators. We also studied the relative efficiency of the semiparametric test the baseline nonparametric test. The theoretical results show that the semiparametric estimator and test are more efficient than the non-parametric estimator and the test.

In the simulation study, we compared five different tests based on samples taken from normal and Gamma distributions respectively. For normal populations, all five tests behave similarly. For samples taken from Gamma distributions, parametric and nonparametric tests, the observed p-values of FVR and NPVR under H_0 are significantly higher than the nominal level meaning that these two tests are inappropriate for testing equality of non-normal population variances.

In terms of sampling design, the power of all proposed semiparametric tests achieves their highest power balanced designs. For the standard non-parametric test, a balanced design may not achieve the best power since the ratio of functions of moments of the unrelated population distributions also impacts the power of the proposed tests.

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